$2d$ Gravity Coupled to Topological Minimal Models

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Abstract

We discuss the properties of genus-0 correlation functions of a topological *minimal* model (the model A_{k+1}) coupled to $2d$ topological gravity. The action of the *twisted* minimal model is perturbed by non-trivial couplings to gravitational descendants, which, in turn, are constructed entirely from the fields in the matter sector of the theory. We develop an explicit formulation, in terms of orthogonal polynomials, for investigating the *large phase space* of the theory. Some useful identities for correlation functions (valid on the *large phase space*) of the theory are established and the *puncture* and *dilaton equations* of topological gravity are obtained as special cases of these general relations. Finally, we obtain an important relation expressing the gravitational couplings in terms of the couplings in the *small phase space*, (*i.e.*, the couplings to the chiral primaries). Thus eventually we are able to solve for the coordinates (couplings) of the large phase space in terms of the LG superpotential characterizing the matter sector of the model.

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1 Introduction

Topological field theories [1] are a class of exactly solvable systems *without* local propagating degrees of freedom. These are, even *without* coupling to gravity, generally *covariant* quantum field theories. It would, however, be incorrect to think that these theories are then of merely mathematical interest, without having any real physical relevance. In fact, in the last few years, we have realized that these theories not only do possess interesting physical properties, but are also related at a more fundamental level to other useful physical theories like the matrix models and $d = 1$ string theory.

One of the handy prescriptions for constructing a large class of useful topological conformal field theories (TCFT) is to start with a $N = 2$ superconformal field theory (SCFT) and then *twist* (or *antitwist* it) it according to some definite set of rules [2], [3] to obtain a TCFT. In particular, if one initially starts with a $0 < c < 3$ *minimal* $N = 2$ model, then one has a TCFT with a finite dimensional physical Hilbert space. It is known that a $N = 2$ SCFT has a special set of states– the so called *chiral primary states*, which form a multiplicative chiral ring. Further when these theories also have a convenient Landau-Ginzburg (LG) description, the chiral ring then becomes the finite-dimensional polynomial (in the LG fields) ring modulo the equation of motion [4, 5, 6]. Under the twisting procedure, it is this chiral ring which becomes identified with the Hilbert space of physical states of the topological theory– *i.e.* all the other states in the original $N = 2$ model *decouple* from the Hilbert space of the topological theory. However, after *coupling* these theories to gravity, this picture changes completely and the Hilbert space of physical states becomes *infinite-dimensional.*

For topological field theories constructed from $N = 2$ superconformal field theories, the underlying chiral ring structure of the latter translates into an *associative* operator algebra for the 3-point correlation function of the topological theory. The 3-point correlation function plays a very fundamental role in topological field theory as all other correlation functions can be determined in terms of it. Thus this associativity constraint provides us, in principle, with sufficient information to obtain all the correlation functions of the theory. If further, the topological theory be perturbed by elements of the chiral ring, the operator algebra still retains the essential structure (even though the chiral ring is then no longer nilpotent). The structure constants of the multiplicative chiral ring algebra then become functions of the perturbing parameters (the couplings) – and their associativity property (which luckily is still retained even in the perturbed theory) once again gives us sufficient constraints to determine the 3-point (and hence in general N-point) correlation functions of the perturbed topological model. However from the practical computational point of view, the above approach does not provide a useful prescription at all. It is at this point that we find that the use of orthogonal polynomials most facilitating [3]. We therefore adopt this approach throughout in our work. We shall try to generalize this method to the case when we couple our model to topological gravity. It turns out that in this case too, the orthogonal polynomials prove to be of invaluable help in investigating the correlation functions and other properties of the gravity-coupled model.

Topological field theories possesss a nilpotent symmetry Q, and the stress-tensor is Q-exact. All novel properties [1, 3] of these theories stem from this characteristic feature. The physical states are the Q-cohomology classes. Further, in this work we have considered as our model of the matter theory, the twisted minimal A_{k+1} model. This has a very convenient representation in terms of a single Landau-Ginzburg field x, and hence the physical Hilbert space will be a finite-dimensional space spanned by the polynomials in x. Topological gravity

is the topological theory associated with the moduli space of Riemann surfaces. In $2d$ topological gravity, one considers a theory with the $2d$ metric $g_{\mu\nu}$ as the dynamical variable, and a classical action given by a vanishing Lagrangian:

$$\mathcal{L}_{classical} \quad = \quad 0$$

The above Lagrangian has more symmetries that the usual diffeomorphism invariance. Gauging these symmetries (by introducing ghost fields and ghosts for ghosts)– then gives us the model of *topological gravity*. We shall consider coupling such a model to topological matter theories. The coupling to topological gravity requires the covariantization of the stress-tensor and the super stress-tensor of the matter sector and also extending our Q symmetry by adding up the contribution from the gravity sector. The resultant $Q_{total} = Q_{top.matter} + Q_{top.gravity}$ then constitutes our nilpotent BRST charge of our $gravity + matter$ topological theory. The physical Hilbert space, which is now the cohomology class of the much enlarged Q_{total} charge – then becomes infinite dimensional. The Q_{total}-cohomology classes will be generically called the *gravitational descendants*. From BRST analysis of the physical spectra [10, 17], we know that the chiral primary fields of the matter sector remain physical observables even after coupling to gravity. They however become *gravitationally dressed* as we shall be seeing in subsequent sections (from eq.(4. 6)). When the topological minimal models are coupled to topological gravity [17], some of the BRST exact states in the matter sector[1] *no longer decouple* from the physical Hilbert space. These states now describe what we shall call the *gravitational descendants*, and will now constitute the much-enlarged (infinite-dimensional) phase space of observables of our theory [8, 9, 16, 18].

From the works of *Losev et. al.* the *remarkable* fact that emerges is that all gravitational descendants can be constructed *entirely* from fields in the matter sector [14, 17]. Further, following *Eguchi et. al.*, we construct these descendant fields by using orthogonal polynomials. We then enlarge the phase space of our minimal theory to an infinite-dimensional one by coupling the minimal action to all such (infinite) gravitational descendants by introducing a coupling for each of these fields. Thus in effect, we treat the minimal topological action in an infinite-dimensional *background* of gravitational descendants, *i.e.* the gravitational descendant (secondaries) sector acts as *external sources* in our formulation. Using our construction for the gravitational descendants, we can obtain useful Ward-identities in the large phase space. It is known that an interesting consequence of the use of orthogonal polynomials in topological field theory (without gravity), is that they enable us to *determine* the exact forms of the couplings characterizing the strengths of the perturbations to the minimal action [3]. One of the main conclusions of our present work is to establish that the above property can be extended even when gravity *is present*. This means that in our general model of *topological matter + topological gravity*, one can still determine, (eventually in terms of the superpotential characterizing the $N = 2$ matter sector) all the couplings to the background fields (consisting of the chiral primaries and the gravitational descendants). Thus the superpotential emerges as possessing *all* the wealth of informations needed to completely investigate such theories. This remarkable fact lends evidence to the assertion that $2d$ gravity is an *induced* effect. Further, if we consider the limit $k \to 0$ of our results, the matter sector becomes trivial and we are then left with a pure gravity theory. Thus in the *limit* when the dynamics of the purely matter sector become *trivial*, we have, from our work, a *topological solution* to the problem of *pure $2d$* gravity.

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[1]By *matter sector*, we shall refer to the twisted Hilbert space of our $N = 2$ models and this consistes of the (left)-chiral primary states only.

The paper is organized as follows. In Sect. 2 we give a brief summary of mostly known results for the case of the perturbed A_{k+1} model (without gravity), and as such this section is essentially a brief recapitulation of results from the existing literature on perturbed Landau-Ginzburg TCFT. Then in Sect. 3, we couple our model to gravity and introduce our constructions for the gravitational descendants. Sect. 4 contains some useful properties for correlation functions and other identities that can be readily derived from our reduction formula obtained in Sect. 3. We also relate and compare our results to the gravity-free case by setting all the gravitational couplings to zero at the end of the day. In the same section, we also obtain explicit expressions for the 2 and 3-point functions for our general gravity-coupled and perturbed model. In Sect. 5, we give some detailed analysis for recursion relations for the correlation functions valid throughout the *large phase space*. We then obtain the generalized puncture and dilaton equations as a special case of these relations. In Sect. 6, we show how we can explicitly determine all the gravitational couplings of the theory in terms of the superpotential characterizing the matter sector. Since the couplings to the chiral primaries are already known, this means that effectively we now know all the couplings in terms of the perturbed superpotential. Thus all information about the *gravity-sector* also stems from the superpotential characterizing the *matter sector*. Hence in a sense, this establishes that topological $2d$ gravity is an *induced* effect (and the case of *pure* gravity simply corresponding to the $k \rightarrow 0$ limit of the matter sector). Finally, in Sect. 7 we summarize our conclusions and comment on the possible future developments. In the appendices, we clarify some of the calculational details used in our work.

2 Solution of the Perturbed A_{k+1} Model Using Orthogonal Polynomials — A Quick Recapitulation

Let us consider the minimal model of the type A_{k+1} corresponding to the *A-* series of the *ADE* classification. This is defined by a single Landau-Ginzburg field x characterized by the superpotential W_0, and the *central charge c* given by:

$$W_0(x) \quad = \quad \frac{x^{k+2}}{k+2}, \qquad c \quad = \quad \frac{3k}{k+2} \tag{2.1}$$

The $(k+1)$ chiral primaries which become the physical operators in the corresponding *twisted* version [2, 3] of the topological theory, are of the form: $x^i = \{1, x, x^2, \dots, x^k\}$.

In the unperturbed theory, the polynomials $\phi_i(x_a) \in \mathcal{R}$ coincide exactly with the basis of the chiral ring[2] and so the 2-point function, on genus-0, can be readily computed using the prescription due to *Vafa* [11] as:

$$\eta_{ij}^{(0)} \quad = \quad < \phi_i(x_a)\phi_j(x_a) > \quad = \quad \oint dx \, \frac{\phi_i(x_a)\phi_j(x_a)}{\partial_x W} \quad = \quad \oint dx \, \frac{x^i x^j}{x^{k+1}}$$
$$= \quad \delta_{i+j,k} \tag{2.2}$$

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[2]And the chiral primaries are identified with $\phi_i = x^i, \quad (i = 0, 1, 2 \dots k)$ and generate the ring $\mathcal{R}$:

$$\phi_i \phi_j = \begin{cases} \phi_{i+j} & i+j \leq k \\ 0 & i+j > k \end{cases}$$

Hence on the sphere the operator ϕ_i is conjugate to ϕ_{k-i}. The 2-point function (in genus-0) defines (the superscript 0 is used to remind us that presently we are dealing with the gravity free case) the *flat* topological metric on the space of the primaries.

2.1 The Perturbed Model

For each of the operators $\phi_i = \{1, x, x^2, x^3, \ldots, x^k\}$, we now introduce couplings $\{t_i\} = \{t_0, t_1, t_2, \ldots, t_k\}$ with $U(1)$ charges (their canonical scaling dimension): $q_{t_i} = \left[1 - \frac{i}{k+2}\right]$, and we consider perturbing the model as:

$$W(x_a; \underline{t}) \quad = \quad \frac{x^{k+2}}{k+2} \quad - \quad \sum_{j=0}^{k} t_j x^j$$

$$\phi_i(x_a) \quad = \quad -\frac{\partial}{\partial t_i} W(x_a; \underline{t}) \quad = \quad x^i$$

where $x^a = \{1, x, x^2, \ldots, x^k\}$, and the notation $\underline{t}$ is presently used as a short form to denote the set $\{t_i\}$ defined above. However this is not the most general form of the perturbation that we can consider, and we demand that our perturbed theory needs to be invariant under *t-reparametrizations*. Thus if we allow t-dependent basis transformations, (such redefinitions like: $t_i \longrightarrow g_i(\underline{t})$, correspond simply to reparametrizations of the couplings)– the metric η_{ij} will obviously lose its nice property of *t-independence*. However, luckily the space of topological field theories is blessed with *zero* curvature[3] and hence there *does* exist a *preferred* (modulo constant t_j-translations) parametrization of the couplings– for which the 2-point function is *constant* (*i.e.* the metric $\eta_{ij}^0 = < \phi_i \phi_j >$ is *constant*, when considered as a function of the moduli t_i). These are the so-called *flat-coordinates* on the space of couplings $\{t_i\}$. We shall therefore always [12] make such a choice of coordinates t_i in our work. The use of orthogonal polynomials in fact provides us with an explicit construction prescription for such flat coordinates in the space of the perturbed theory.

Armed with the above knowledge, we may, more generally, consider perturbing the potential as:

$$W(x_a; \underline{t}) \quad = \quad \frac{x^{k+2}}{k+2} \quad - \quad \sum_{j=0}^{k} g_j(\underline{t}) \, x^j \tag{2. 3}$$

where $g_j(\underline{t})$ are *a priori* arbitrary functions of $\{t_i\}$, and the polynomials $\phi_i(x_a; \underline{t})$

$$\phi_i(x_a; \underline{t}) \quad \stackrel{\text{def}}{=} \quad -\partial_i W(x_a; \underline{t}), \qquad (\text{where} \quad \partial_i = \frac{\partial}{\partial t_i}) \tag{2. 4}$$

now generate the chiral ring *via* the multiplication rule:

$$\phi_i(x_a; \underline{t}) \, \phi_j(x_a; \underline{t}) = \sum_l \mathcal{C}_{ij}{}^l(\underline{t}) \, \phi_l(x_a; \underline{t}) \qquad (\text{mod} \quad \frac{\partial W(x_a; \underline{t})}{\partial x_b})$$

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[3] One can, in fact, set up a covariant formulation in the space of the perturbing coupling constants, in which the ordinary derivatives $\frac{\partial}{\partial t_j}$ are replaced by covariant ones with respect to the couplings. It then transpires that the Christoffel symbols in these covariant derivatives are the necessary *contact* terms, that arise in constructing the theory with a general choice of operators.

with the perturbed structure constants $\mathcal{C}_{ij}{}^l(\underline{t})$ still satisfying the same associativity constraints:

$$\sum_m \mathcal{C}_{ij}{}^m(\underline{t})\,\mathcal{C}_{mkl}(\underline{t}) = \sum_m \mathcal{C}_{ik}{}^m(\underline{t})\,\mathcal{C}_{mjl}(\underline{t})$$

From the general properties of the correlation functions, (Ward identities), we know that the 2-point function (which also defines the metric) remains unaltered even in the presence of perturbations (*i.e.*, is $\underline{t}$- independent). Hence the 2-point function defined by $\phi_i(x_a;\underline{t})$ must *coincide* with that defined by $\phi_i(x_a)$. Hence we have the identification:

$$\begin{aligned}
\eta_{ij}^{(0)}(\underline{t}) \quad &= \quad <\phi_i(x_a;\underline{t})\,\phi_j(x_a;\underline{t})> \quad = \quad <\phi_i(x_a)\,\phi_j(x_a)> \\
&= \quad \eta_{ij}^{(0)}(0) \quad = \quad \delta_{i+j,k}
\end{aligned} \tag{2. 5}$$

2.2   Orthogonal Polynomials

Let us now introduce a *generating function* $L_0(x_a;\underline{t})$, (once again, the subscript zero is used to remind us that currently, we are dealing with the gravity-free case) defined by:

$$W(x_a;\underline{t}) \quad \overset{\text{def}}{=} \quad \frac{L_0^{k+2}(x_a;\underline{t})}{k+2} \quad = \quad \frac{x^{k+2}}{k+2} \quad - \quad \sum_{j=0}^{k} g_j(\underline{t})\,x^j \tag{2. 6}$$

We can then define the $(k+1)$ *orthogonal polynomials* $\Phi_i(x_a;\underline{t})$ by:

$$\Phi_i(x_a;\underline{t}) \quad \overset{\text{def}}{=} \quad \frac{1}{i+1}\left[\frac{\partial}{\partial x}L_0^{i+1}\right]_+ \qquad (i = 0,1,2,\ldots,k.) \tag{2. 7}$$

where the subscript $+$ indicates a truncation of the series to only positive powers of x. These polynomials are *orthogonal* with respect to the definition of the inner product in the ring $\mathcal{R}$. One immediate consequence of the above definition is:

$$\Phi_0(x_a;\underline{t}) \quad = \quad \mathbf{1}$$

Then it is not difficult to establish the *identification*:

$$\Phi_j(x_a;\underline{t}) \quad \equiv \quad \phi_j(x_a;\underline{t}) \tag{2. 8}$$

The fundamental correlation function in the perturbed theory known to be the 3-point function $-<\phi_i(x_a;\underline{t})\phi_j(x_a;\underline{t})\phi_l(x_a;\underline{t})>$. Setting one of the indices to zero, then defines the metric of the theory, $<\phi_i(x_a;\underline{t})\phi_j(x_a;\underline{t})\Phi_0(x_a;\underline{t})> \quad = \quad \eta_{ij}^{(0)}(\underline{t})$ But in the absence of gravity, $\phi_0(x_a;\underline{t}) = \mathbf{1}$, and the above expresion *reduces* to the expression for the 2-point function, *i.e.* $<\phi_i(x_a;\underline{t})\phi_j(x_a;\underline{t})> \quad = \quad \eta_{ij}^{(0)}(\underline{t})$. The *flat-coordinates* (the couplings) are then known to be given by:

$$t_{k-i} \quad = \quad -\frac{1}{i+1}\left(\text{res}\,L_0^{i+1}\right) \tag{2. 9}$$

Once again the derivation of the above result relies on the *t-independence* properties of the 2-point function of the theory.

Note: The analogous result for the couplings for the chiral primaries with gravity *switched on* is (for an explanation of the notations used please see the next section):

$$t_{0,k-i} \quad = \quad -\frac{1}{i+1}\left(\text{res}\,L^{i+1}\right) \quad = \quad -\frac{(k+2)^{\frac{i+1}{k+2}}}{i+1}\,\text{res}\left(W^{\frac{i+1}{k+2}}\right) \qquad \text{for } 0 \le i \le k \tag{2. 10}$$

3 The Perturbed and Gravity-Coupled Model

We shall now consider coupling our topological matter theory (corresponding to the 'twisted' version of the A_{k+1} model) to topological gravity. The primary fields in the *purely* matter sector will still be the chiral primary fields ϕ_i ($i = 0, 1, 2, \ldots, k$), (and are in one-to-one correspondence with the elements of the perturbed chiral ring of the $N = 2$ model) and after coupling to gravity, the complete[4] spectrum of physical operators of our theory will be given by the above chiral primaries together with all (can be infinite in number) their *gravitational descendants* $\{\sigma_N(\phi_i)\}$. These $\{\sigma_N(\phi_i)\}$ are the BRST invariant operators with respect to the total BRST charge of the gravity-coupled model.

In a free-field formulation of pure topological gravity, the (super) conformal gauge fixed action consists of the following set of dynamical fields: the Liouville field ρ and its superpartner ψ, the associated anti-ghosts π and χ, as well as the usual spin $(2, -1)$ Faddeev-Popov (b, c) ghost fields and their superpartners (β, γ). The fundamental BRST invariant operator comes from the ghost sector and is

$$\gamma_0 \quad = \quad \frac{1}{2}\left(\partial\gamma + \gamma\partial\beta - c\partial\psi\right) \ - \ c.c$$

and the corresponding complete set of non-trivial physical observables in the gravity-sector are the family of operators defined by:

$$\sigma_N \quad = \quad \gamma_0{}^N \cdot \mathbf{P} \qquad \text{where} \quad \mathbf{P} \ = \ c\,\overline{c}\,\delta(\gamma)\,\delta(\overline{\gamma}) \qquad (N = 0, 1, 2, \ldots) \qquad (3.\ 1)$$

and $\mathbf{P}$ is the *puncture* operator. After coupling the above theory to a topological matter system, the complete spectrum of (total) BRST invariant operators of the coupled system are given by:

$$\{\sigma_{N,i}\} \quad \equiv \quad \{\sigma_N(\phi_i)\} \quad = \quad \phi_i \cdot \gamma_0{}^N \cdot \mathbf{P} \qquad (i = 0, 1, 2, \ldots k)$$

where ϕ_i are the left (or right) moving chiral primary fields from the matter sector. These fields $\sigma_{N,i}$ are our gravitational descendants and their constrction and properties have been discussed in details in the existing literature [8, 9, 14, 16, 18] and so we do not elaborate on these issues here. The operators which are the gravitational descendants of the *identity* operator are those that are present in a model of pure topological gravity (*i.e.*, before any coupling to matter models).

It is known [14] that these descendants of the chiral primaries can be constructed *entirely* from fields in the matter sector (which may consist of the twisted version of some $N = 2$ theory, or topological sigma model). We shall further call the special operator $\sigma_0(\phi_0) = \phi_0 = \mathbf{P}$ the *puncture operator* [18] and this operator will play a vital role in our future discussions.

With hindsight, we may make the following observation here. As in the gravity-free case studied in the earlier section, the fundamental correlation function in the small phase space (defined later) is once again the 3-point function: $\quad < \phi_i(x_a; \underline{t})\phi_j(x_a; \underline{t})\phi_l(x_a; \underline{t}) >$. Setting one of its indices to zero then defines the *metric* of the theory:

$$\eta_{ij} \quad \overset{\text{def}}{=} \quad < \phi_i(x_a; \underline{t})\phi_j(x_a; \underline{t})\phi_0(x_a; \underline{t}) > \quad = \quad < \phi_i(x_a; \underline{t})\phi_j(x_a; \underline{t})\,\mathbf{P} > \qquad (3.\ 2)$$

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[4]It is believed, though it has not yet been proved rigorously that the complete set of physical operators in *topological gravity* + *topological matter* does consist of the set $\{\sigma_N(\phi_i)\}$. In minimal topological 2d gravity, there are operators σ_N, $N = 0, 1, 2, \ldots$, of *ghost number* $(2N - 2)$ which correspond to $(2N - 2)$-dimensional submanifold of moduli space.

However with non-vanishing coupling to the gravitational descendants, $\mathbf{P} \neq \mathbf{1}$, and hence in this case (quite unlike the gravity-free case), this does not coincide with the 2-point function. Thus *pure topological gravity* is the case in which the only primary field in the theory is the *puncture operator* and in this case the complete spectrum of physical states of the theory consists of: $\{\phi_0 = \mathbf{P}$, together with all its gravitational descendants $\sigma_N(\mathbf{P})\}$.

Introducing the infinite set of couplings (the space of *all* couplings $\{t_{N,i}\}$ will be called the *phase space* of the theory) $\underline{t} = \{t_{N,i}\}$, coupling to the $\sigma_N(\phi_i)$'s, where $N = 0, 1, 2, \ldots, \infty$, and $i = 0, 1, 2, \ldots, k$, our generic action will look like:

$$\mathcal{S} \quad = \quad \mathcal{S}_0 - \sum_{N,j} t_{N,j} \int \sigma_N(\phi_j) \tag{3. 3}$$

where $\mathcal{S}_0$ is the *minimal* gravitational action *plus* the action of the topological A_{k+1} model, *i.e.* $\mathcal{S}_0 = \mathcal{S}_{N=2} + \mathcal{S}_{top.grv}$. We are thus studying the A_{k+1} topological model (on the sphere) in the presence of an infinite-dimensional background of gravitational descendant fields which we treat as external sources with relative strengths determined by the couplings $t_{N,i}$. The *finite* dimensional phase space with $t_{N,i} = 0$, for $N > 0$ (which is the same as $\{t_{\widetilde{N},i}\} = 0$) plays a very important role in our theory and we shall call this the *small phase space*. Thus the *small phase space* is a $(k+1)$- dimensional phase space with affine coordinates $t_{0,i}$. The *small phase space* thus describes the moduli space of topological field theories that can be reached by (relevant and marginal) perturbations of the minimal models. The operator $\sigma_0(\phi_0) = \phi_0 = \mathbf{P}$ is the *puncture operator*, and its coupling t_0 plays the role of the cosmological constant. The operator σ_1 is called the *dilaton operator* (borrowing terminology from string theory). The puncture operator produces the crucially important contact terms when inserted in correlation functions.

In terms of the superpotential characterizing the theory this translates into considering a perturbed superpotential of the form:

$$W(x_a; \underline{t}) \quad = \quad \frac{x^{k+2}}{k+2} - \sum_{N=0}^{\infty} \sum_{i=0}^{k} t_{N,i} \, \sigma_N(\phi_i) \tag{3. 4}$$

where $\sigma_N(\phi_i)$ is the Nth *gravitational descendant* of the chiral primary field $\phi_i(x; \underline{t})$,

$$\sigma_N(\phi_i) \quad = \quad -\frac{\partial W}{\partial t_{N,i}} \tag{3. 5}$$

$$\phi_i \quad = \quad -\frac{\partial W}{\partial t_{0,i}} \quad = \quad -\frac{\partial W}{\partial t_i} \tag{3. 6}$$

Notation used :   Let us first try to clarify the *notation* that we are going to use in our work.

1. Except when otherwise mentioned the subscript indices $i, N, \widetilde{N}$ can run over the following range of values:

$$i \quad = \quad 0, 1, 2, \ldots, k.$$
$$N \quad = \quad 0, 1, 2, \ldots, \infty$$
$$\widetilde{N} \quad = \quad 1, 2, 3, \ldots, \infty$$

2. The quantities $\underline{t}$ will henceforth refer to the complete set of coupling parameters for the *perturbed* and *gravity-coupled* model. Thus

$$
\begin{aligned}
\underline{t} &= \{t_{N,i}\} \qquad\qquad (N = 0,1,2,\ldots,\infty\,;\, i = 0,1,2,\ldots,k) \\
&= \{t_0, t_1, t_2, \ldots, t_k, t_{1,0}, t_{1,1}, t_{1,2}, \ldots, t_{1,k}, \ldots, t_{N,0}, t_{N,1}, t_{N,2}, \ldots, t_{N,k}, \ldots\}
\end{aligned}
$$

Allowing arbitrary reparametrizations in the coupling constants space (the *large* phase space) $\{t_{N,i}\}$, $N = 0,1,2,3,\ldots$, and $i = 0,1,2,\ldots,k$, the most general form of the perturbed and gravity-coupled superpotential is:

$$
W(x_a;\underline{t}) \quad = \quad \frac{x^{k+2}}{k+2} - \sum_{N=0}^{\infty} \sum_{i=0}^{k} g_{N,i}(\underline{t})\, \sigma_N(\phi_i) \tag{3. 7}
$$

where $g_{N,i}(\underline{t})$ are once again a priori arbitrary functions of the couplings $\underline{t}$. The *generating function* $L(x_a,\underline{t})$ is now defined by:

$$
W(x_a;\underline{t}) \quad \overset{\text{def}}{=} \quad \frac{L^{k+2}(x_a;\underline{t})}{k+2} \quad = \quad \frac{x^{k+2}}{k+2} - \sum_{N=0}^{\infty} \sum_{i=0}^{k} g_{N,i}(\underline{t})\, \sigma_N(\phi_i) \tag{3. 8}
$$

Then the Nth gravitational descendant $\sigma_N(\phi_i)$ of the chiral primary field ϕ_i is given by:

$$
\sigma_N(\phi_i) \quad \overset{\text{def}}{=} \quad \frac{b(N,i)}{a(N)+i+1}\left[\partial_x L^{a(N)+i+1}\right]_+ \tag{3. 9}
$$

Further, we also have the identification: $\sigma_0(\phi_i) = \phi_i$, $\forall\, i \quad 0 \le i \le k$. From consistency requirements: $\sigma_N(\phi_i) \equiv 0$, for $N < 0$, and $\forall\, i$. and we further demand that the as yet arbitrary numerical functions $a(N)$ and $b(N,i)$ also satisfy the constraints[5] :

$$
\begin{aligned}
a(N) - a(N-1) &= (k+2) \\
\left[\frac{b(N,i)}{a(N)+i+1}\right] &= b(N+1,i)
\end{aligned} \tag{3. 11}
$$

Note: Using the definitions (3. 6) and (3. 9) we can readily obtain the equation:

$$
-\frac{\partial W}{\partial t_i} \quad = \quad \frac{(k+2)^{\frac{i+1}{k+2}}}{(i+1)}\left[\partial_x W^{\frac{i+1}{k+2}}\right]_+
$$

So that *relabelling* the couplings $t_i \longrightarrow \tau_i$, where

$$
\tau_{j-1} \quad \overset{\text{def}}{=} \quad -(N)^{-\frac{j}{N}} j\, t_{j-1} \qquad\qquad (\text{where} \quad N = k+2)
$$

[5]The above constraints may be readily solved to give:

$$
\begin{aligned}
a(N) &= N(k+2) \\
b(N,i) &= (k+2)^{-N}\left[\frac{\Gamma(\lambda)}{\Gamma(\lambda+N)}\right] \\
&\qquad \left(N = 0,1,2,\ldots\infty;\ \text{and}\ \lambda = \left(\frac{i+1}{k+2}\right)\right)
\end{aligned} \tag{3. 10}
$$

we can rewrite the above equation in the form:

$$\frac{\partial}{\partial \tau_p} W = \left[\partial_x W^{p/N}\right]_+ \qquad (\text{for} \quad p = 1, 2, ..., N-1.) \qquad (3.\ 12)$$

which is essentially similar to the equation for the p-th primary KdV flows.

The correlation function $< \sigma_N(\phi_i)\phi_j >$ can be evaluated by using the prescription [11] due to *Vafa* and we eventually get a useful reduction formula due to *Eguchi et. al.* [10] as follows:

$$\sigma_N(\phi_i) = W' \int^x \sigma_{N-1}(\phi_i) + \sum_{j=0}^{k} \phi_{k-j}\left(\frac{\partial}{\partial t_j} \mathbb{R}^i_{(N)}(\underline{t})\right) \qquad (3.\ 13)$$

where we have defined:

$$\mathbb{R}^i_{(M)}(\underline{t}) \overset{\text{def}}{=} -b(M+1,i) \oint dx\, L^{M(k+2)+i+1} \qquad (3.\ 14)$$

The quantity $\mathbb{R}^i_{(M)}(\underline{t})$ may be identified with the Gelfand-Dikki potential of the KdV hierarchy. From the above definition, we can at once draw the folllowing inferences.

1. Clearly as a special case (setting all gravitational couplings to zero) we see:

$$\mathbb{R}^i_{(0)}(\underline{t}) = -\left(\frac{1}{i+1}\right)\oint dx\, L^{i+1}$$

$$\implies \mathbb{R}^i_{(0)}(\underline{t}) = t_{0,k-i} \qquad (3.\ 15)$$

$$\text{Similarly} \qquad \mathbb{R}^i_{(1)}(\underline{t}) = -\frac{1}{(i+1)(k+i+3)}\oint dx\, L^{k+i+3}$$

$$\implies \mathbb{R}^i_{(1)}(\underline{t})\Big|_{\{t_{\widetilde{N},i}\}=0} = -\frac{\partial \mathcal{F}_0}{\partial t_i} \qquad (3.\ 16)$$

where $\mathcal{F}_0(t_0, t_1, \ldots, t_k)$ is the free-enegy of the model in the absence of gravity.

2. From the definition, we also see that

$$\mathbb{R}^i_{(N)}(\underline{t}) = 0 \qquad \text{for } N < 0. \qquad (3.\ 17)$$

Further we can also establish the relation

$$(-1)^r \frac{\partial^r}{\partial t_0{}^r} \mathbb{R}^l_{(N)}(\underline{t})\Big|_{\{t_{\widetilde{N},i}\}=0} = \mathbb{R}^l_{(N-r)}(\underline{t}) \qquad (3.\ 18)$$

for the quantity $\mathbb{R}^i_{(N)}(\underline{t})$ by putting all the couplings to the gravitational descendants to zero at the end of the day.

Thus we have the very important *reduction formula for the gravitational descendants*:

$$\forall\ \boldsymbol{N > 0}, \qquad \boldsymbol{\sigma_N(\phi_i)} = \boldsymbol{W'}\int^{\boldsymbol{x}} \boldsymbol{\sigma_{N-1}(\phi_i)} + \sum_{\boldsymbol{j=0}}^{\boldsymbol{k}} \boldsymbol{\phi_{k-j}}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial t_j}} \boldsymbol{\mathbb{R}^i_{(N)}(\underline{t})}\right)$$

$$\text{while for}\ \ \boldsymbol{N = 0}, \quad \boldsymbol{\sigma_0(\phi_i)} = \boldsymbol{\phi_i} \qquad \boldsymbol{\forall\ i,\ 0 \le i \le k} \qquad (3.\ 19)$$
$$\text{and for}\ \ \boldsymbol{N < 0}, \quad \boldsymbol{\sigma_N(\phi_i)} = \boldsymbol{0} \qquad \boldsymbol{\forall\ i,\ 0 \le i \le k}$$

Clearly from the above reduction formula, we get for $N > 0$, the useful identification:

$$< \sigma_N(\phi_i)\phi_j > \quad = \quad \frac{\partial}{\partial t_j} \mathbb{R}^i_{(N)}(\underline{t}) \qquad (3.\ 20)$$

Further, inserting the above result into the reduction formula for the gravitational descendants, we get the following alternatve form:

$$\forall\ N > 0, \quad \sigma_N(\phi_i) \quad = \quad W'\int^x \sigma_{N-1}(\phi_i) + \sum_{j=0}^{k} \phi_{k-j} < \sigma_N(\phi_i)\,\phi_j > \qquad (3.\ 21)$$

4   Some important properties and results

In this section, we shall try to obtain some useful conclusions which follow readily from our constructions. Some of these results (like the *contact algebra*) are already known in the context of discussions of $2d$ topological gravity based on the path-integral or other approaches. We shall rederive these results and also extend and generalize some of them. We will be able to obtain an interesting hierarchy of differential equations (which are supposed to be the generalization of the multi-contact term algebra). We shall also identify an *operator correspondence* which, we expect, does have deeper mathematical implications as similar operations also appear in an entirely different context (in the mathematical works of *Saito* on *higher residue pairing*). Finally we shall obtain an important recursive differential equation for the $\mathbb{R}^l_{(N)}(\underline{t})$, which incorporates the multiplicative chiral ring property of the primary fields of the theory. This relation is reminiscent of the *flatness* criterion on the space of couplings.

4.1   Contact Algebra

Let us now try to obtain some equations (in the large phase space) which follow directly from the reduction formula for the gravitational descendants. In the analysis of $2d$ gravity (from other standpoints), it is well known that there are *contact terms* in the expressions for the correlation functions. We shall now obtain the equivalent expressions for these contact terms based on our approach.

1. From the expression:

$$\left(\frac{\sigma_N(\phi_i)}{W'}\right) \quad = \quad \int^x \sigma_{N-1}(\phi_i) \quad + \quad \sum_{j=0}^{k} \left(\frac{\phi_{k-j}}{W'}\right), \left(\frac{\partial}{\partial t_j} \mathbb{R}^i_{(N)}(\underline{t})\right)$$

we have the important result:

$$\partial_x\left[\left(\frac{\sigma_N(\phi_i)}{W'}\right)\right]_+ \quad = \quad \sigma_{N-1}(\phi_i) \qquad (4.\ 1)$$

Thus setting $N = 0$ in the above equation we get:

$$\partial_x\left[\left(\frac{\sigma_0(\phi_i)}{W'}\right)\right]_+ \equiv \partial_x\left[\left(\frac{\phi_i}{W'}\right)\right]_+ = 0 \qquad (4.\ 2)$$

Using the definitions it is quite simple to derive the following set of first-order differential equations governing the $t_{N,i}$ dependence of the descendant fields $\sigma_N(\phi_i)$

$$\frac{\partial\,\sigma_N(\phi_i)}{\partial t_{M,j}} = \frac{\partial\,\sigma_M(\phi_j)}{\partial t_{N,i}} = -\partial_x\left[\left(\frac{\sigma_N(\phi_i)\,\sigma_M(\phi_j)}{W'}\right)\right]_+ \qquad (4.\ 3)$$

These equations are *integrable*, and in fact their integrability may be verified by noting that an *explicit* solution to the above system is given by:

$$\sigma_N(\phi_i) \quad \sim \quad -\frac{1}{N(k+2)+i}\left[(-W)^{N(k+2)+i}\right]'_+ \qquad (4.\ 4)$$

where the primes refer to derivatives with respect to x.

An operator $\sigma_N(\phi_i)$ inserted in a correlation function creates a puncture and thus introduces a moduli that must then be integrated over. Thus in effect $\sigma_N(\phi_i)$ is represented by an integration of its 2-form partner. This integration receives a special contribution whenever $\sigma_N(\phi_i)$ approaches some other operator $\sigma_M(\phi_j)$ on the surface. This special contribution is what we call the contact term. Thus in the language of $2d$ gravity the term $C(N, i; M, j)$ is identified as a *contact* term created by the *collision* of two gravitational descendant operators, where

$$C(N, i; M, j) \quad \stackrel{\text{def}}{=} \quad \left[\frac{\sigma_N(\phi_i)\sigma_M(\phi_j)}{W'}\right]'_+$$

These contact terms are of great significance in obtaining the expressions for the N-point $(N > 3)$ correlation functions of the theory as it is the presence of these contact terms that render these generic N-point $(N > 3)$ correlation functions *symmetric* under the interchange of of its arguments. The contact term $C(N, i; M, j)$ expresses the gravitational *dressing* of the descendant field $\sigma_N(\phi_i)$ by the couplings $t_{M,j}$ and we note the nice *duality* that this is the same as the gravitational *dressing* of the descendant field $\sigma_M(\phi_j)$ by the couplings $t_{N,i}$.

From (4. 3), we may conclude the following by setting different specific values of N, M:

$\boxed{M = N = 0}$:   In this case, we get

$$\frac{\partial\,\sigma_0(\phi_i)}{\partial t_{0,j}} = -\partial_x\left[\left(\frac{\sigma_0(\phi_i)\,\sigma_0(\phi_j)}{W'}\right)\right]_+$$

$$i.e., \quad \frac{\partial\,\phi_i}{\partial t_j} = -\partial_x\left[\left(\frac{\phi_i\,\phi_j}{W'}\right)\right]_+ \qquad (4.\ 5)$$

exactly the same as is known in the gravity-free case. Further, the above equation is of great significance as it imposes the condition of *flatness* on the perturbing coordinates—the flat-coordinates are thus the solutions to the above equation.

$\boxed{M = 0}$: In this case, we get

$$\frac{\partial\,\sigma_N(\phi_i)}{\partial t_j} = \frac{\partial\,\phi_j}{\partial t_{N,i}} = -\partial_x\left[\left(\frac{\sigma_N(\phi_i)\,\phi_j}{W'}\right)\right]_+$$

which tells us that:

$$\frac{\partial\,\phi_j}{\partial t_{N,i}} = -\partial_x\left[\left(\frac{\sigma_N(\phi_i)\,\phi_j}{W'}\right)\right]_+ \tag{4. 6}$$

$$\neq \quad 0, \qquad \text{in general.}$$

Thus *the chiral primaries acquire non-trivial dependences (at higher order) on the gravitational couplings – that is they get 'gravitationally dressed'.* As a consequence, the correlation functions involving the chiral primaries will *depend* on the couplings in the large phase space [3].

Setting $j = 0$ in the above eq.(4. 6), and using the result (4. 1) gives we get the important relation:

$$\frac{\partial\,\sigma_N(\phi_i)}{\partial t_0} = -\,\sigma_{N-1}(\phi_i) \tag{4. 7}$$

We may note here that the operator t_0 couples to the *puncture operator* in our expression for the perturbed action. In terms of path-integrals, the operator $\left(-\frac{\partial}{\partial t_0}\right)$ correspondes to **P** operator insertions, and hence the above equation gives us an useful reduction formula for expressing correlation functions of Nth- gravitational descendant in terms of lower (*i.e.,* $(N-1)^{th}$ and so on) descendant fields.

2. Using the above relations recursively we can also obtain the following important hierarchy of equations:

$$\frac{\partial\,\sigma_N(\phi_i)}{\partial t_{M,j}} = -\partial_x\left[\left(\frac{\sigma_N(\phi_i)\,\sigma_M(\phi_j)}{W'}\right)\right]_+$$

$$\frac{\partial^2\,\sigma_N(\phi_i)}{\partial t_{M,j}\,\partial t_{L,s}} = \partial_x^2\left[\left(\frac{\sigma_N(\phi_i)\,\sigma_L(\phi_s)\,\sigma_M(\phi_j)}{W'^2}\right)\right]_+$$

$$\frac{\partial^3\,\sigma_N(\phi_i)}{\partial t_{M,j}\,\partial t_{L,s}\,\partial t_{P,l}} = -\partial_x^3\left[\left(\frac{\sigma_N(\phi_i)\,\sigma_L(\phi_s)\,\sigma_P(\phi_l)\,\sigma_M(\phi_j)}{W'^3}\right)\right]_+ \tag{4. 8}$$

$$\vdots$$
$$\vdots$$
$$\vdots$$

Or *equivalently* the hierarchy:

$$\frac{\partial^2\,W}{\partial t_{M,j}\,\partial t_{N,i}} = \partial_x\left[\left(\frac{\sigma_N(\phi_i)\,\sigma_M(\phi_j)}{W'}\right)\right]_+$$

$$\frac{\partial^3\,W}{\partial t_{M,j}\,\partial t_{L,s}\,\partial t_{N,i}} = -\partial_x^2\left[\left(\frac{\sigma_N(\phi_i)\,\sigma_L(\phi_s)\,\sigma_M(\phi_j)}{W'^2}\right)\right]_+$$

$$\frac{\partial^4\,W}{\partial t_{M,j}\,\partial t_{L,s}\,\partial t_{P,l}\,\partial t_{N,i}} = \partial_x^3\left[\left(\frac{\sigma_N(\phi_i)\,\sigma_L(\phi_s)\,\sigma_P(\phi_l)\,\sigma_M(\phi_j)}{W'^3}\right)\right]_+ \tag{4. 9}$$

$$\vdots$$
$$\vdots$$
$$\vdots$$

Looking at the above sequence of non-linear differential equations, we can can make the following observation about the *operator equivalence*, which may be symbolically expressed as:

$$\prod_{p=1}^{n+1} \left(\frac{\partial}{\partial t_{M,i_p}} \right) \quad \longleftrightarrow \quad (-1)^{n+1} \partial_x{}^n \left[\frac{1}{W'^n} (\cdots) \right]_+ \qquad (n = 0, 1, \ldots, k) \qquad (4.\ 10)$$

We think that the above observation might have deeper mathematical implications in terms of Saito's *higher residue pairing* [10, 14, 15], but this needs to be investigated further.

4.2 Recursive Property for $R^l_{(N)}(\underline{t})$

We shall now obtain an interesting property for the $\mathbb{R}^l_{(N)}(\underline{t})$ by exploiting the chiral ring structure of the matter sector of the theory. We shall obtain a recursive second-order differential equation with respect to the parameters in the small phase space for $\mathbb{R}^l_{(N)}(\underline{t})$. It is interesting to note that in this equation all the differential operators act only on the small phase space. We shall see in a subsequent section that all the coordinates in the large phase space can be eventually expressed in terms of those on the small phase space, and thus such a relation is not totally unexpected. We shall evoke the multiplicative properties of the chiral ring of the primary fiels in the spectrum of the theory (the ring structure is still preserved even though it is no longer a nilpotent ring) to obtain this desired relation.

Beginning from the definition (3. 14) we have:

$$
\begin{aligned}
\frac{\partial^2}{\partial t_i \partial t_j} \mathbb{R}^l_{(N)}(\underline{t}) \quad &= \quad -b(N-1,l) \oint dx\, L^{(N-2)(k+2)+l+1} \left(\phi_i \phi_j \right) \\
&\qquad + b(N,l) \oint dx\, L^{(N-1)(k+2)+l+1} \left(\frac{\partial \phi_i}{\partial t_j} \right) \\
&= \quad -\sum_m \mathcal{C}_{ij}{}^m \left(\frac{\partial}{\partial t_m} \mathbb{R}^l_{(N-1)}(\underline{t}) \right) \\
&\qquad + b(N,l) \oint dx\, L^{(N-1)(k+2)+l+1} \left(\frac{\partial \phi_i}{\partial t_j} \right)
\end{aligned}
$$

where we have used the multiplicative chiral ring algebra in the form:

$$\phi_i \, \phi_j \quad = \quad \sum_m \mathcal{C}_{ij}{}^m \, \phi_m$$

with $\mathcal{C}_{ij}{}^m \equiv \mathcal{C}_{ij}{}^m(t_0, t_1, t_2, \ldots, t_k)$ being the structure constants of the chiral ring algebra of our $N = 2$, A_{k+1} model.

Thus we may write the above equation in the form:

$$b(N,l) \oint dx\, L^{(N-1)(k+2)+l+1} \left(\frac{\partial \phi_i}{\partial t_j} \right) = \frac{\partial^2}{\partial t_i \partial t_j} \mathbb{R}^l_{(N)}(\underline{t}) + \sum_m \mathcal{C}_{ij}{}^m \left(\frac{\partial}{\partial t_m} \mathbb{R}^l_{(N-1)}(\underline{t}) \right) \quad (4.\ 11)$$

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An interesting fact of life is that the left-hand side expression of the above equation may be shown actually to *vanish* exactly (A. 1) (for proof see the appendix), *i.e.*,

$$\oint dx \, L^{(N-1)(k+2)+l+1}\left(\frac{\partial \phi_i}{\partial t_j}\right) \quad = \quad 0 \tag{4. 12}$$

Leaving us with the following important property:

$$\frac{\partial^2}{\partial t_i \partial t_j}\mathbb{R}^l_{(N)}(\underline{t}) \; + \; \sum_m \mathcal{C}_{ij}{}^m\left(\frac{\partial}{\partial t_m}\mathbb{R}^l_{(N-1)}(\underline{t})\right) \quad = \quad 0 \tag{4. 13}$$

The above gives us an useful recursive, second-order differential equation for the $\mathbb{R}^l_{(N)}(\underline{t})$, which also introduces the structure constants of the chiral ring into the picture. Let us then analyze some special cases of the above equation.

Before that in passing, we can make the following interesting *observation* here. Let us *define* a quantity $\Psi^{(l)}(\underline{t})$ by

$$\Psi^{(l)}(\underline{t}) \quad \overset{\text{def}}{=} \quad \sum_{N=-\infty}^{N=\infty} \mathbb{R}^l_{(N)}(\underline{t}) \quad = \quad \sum_{N=0}^{N=\infty} \mathbb{R}^l_{(N)}(\underline{t}) \tag{4. 14}$$

the equality of the two summations being obvious because $\mathbb{R}^l_{(N)}(\underline{t}) = 0$, for $N < 0$. Now if we sum both sides of the equation (4. 13) over $N = -\infty$ to ∞, we have the following relation for $\Psi^{(l)}(\underline{t})$:

$$\left[\frac{\partial^2}{\partial t_i \partial t_j} \; + \; \sum_m \mathcal{C}_{ij}{}^m\left(\frac{\partial}{\partial t_m}\right)\right]\Psi^{(l)}(\underline{t}) \quad = \quad 0 \tag{4. 15}$$

The differential operator within the square braces above can be easily identified with the familiar covariant derivatives and the above equation can be interpreted as a *flatness* criterion in the small phase space for the quantity $\Psi^{(l)}(\underline{t})$.

Special case: If we set all *gravitational couplings to zero*, as a special case, we see that the above equation may be simplified further as follows. Using the equation (5. 5) (proved later) $\frac{\partial}{\partial t_0}\mathbb{R}^l_{(N)}(\underline{t}) \; = \; -\mathbb{R}^l_{(N-1)}(\underline{t})$ in the gravity-free case, we can also rewrite the above equation in the alternative form:

$$\boldsymbol{\nabla}_{ij}\,\mathbb{R}^l_{(N)}(\underline{t})\Big|_{\{t_{\widetilde{N},i}\}=0} \quad = \quad \mathbf{0} \tag{4. 16}$$

$$\text{where} \quad \boldsymbol{\nabla}_{ij} \quad \equiv \quad \boldsymbol{\nabla}_{ij}(t_0, t_1, t_2, \ldots, t_k)$$

$$\overset{\text{def}}{=} \quad \left[\frac{\partial^2}{\partial t_i \partial t_j} - \sum_{m=0}^{k} \mathcal{C}_{ij}{}^m \frac{\partial^2}{\partial t_0 \partial t_m}\right]$$

Corollary: Setting $N = 1$ and also all gravitational couplings to zero at the end of the day, in the previous equation (4. 13) we get :

$$\frac{\partial^2}{\partial t_i \partial t_j}\mathbb{R}^l_{(1)}(\underline{t}) \; + \; \sum_m \mathcal{C}_{ij}{}^m\left(\frac{\partial}{\partial t_m}\mathbb{R}^l_{(0)}(\underline{t})\right) \quad = \quad 0$$

$$\implies \quad \frac{\partial^2}{\partial t_i \partial t_j} \mathbb{R}^l_{(1)}(\underline{t}) \Big|_{\{t_{\widetilde{N},i}\}=0} + \sum_m \mathcal{C}_{ij}{}^m \left(\frac{\partial}{\partial t_m} \mathbb{R}^l_{(0)}(\underline{t}) \right) \Big|_{\{t_{\widetilde{N},i}\}=0} = 0$$

$$\text{Hence,} \quad \frac{\partial^2}{\partial t_i \partial t_j} \mathbb{R}^l_{(1)}(\underline{t}) \Big|_{\{t_{\widetilde{N},i}\}=0} = -\mathcal{C}_{ij}{}^{k-l}$$

$$= -c_{ij}{}^l$$

where $c_{ij}{}^l = c_{ij}{}^l(t_0, t_2, \dots, t_k)$ is related to the 3-point correlation function of the *gravity-free* model via the rule: $c_{ij}{}^l = c_{ijm} \eta_{(0)}^{lm}$. The above result also confirms our previous identification:

$$\mathbb{R}^i_{(1)}(\underline{t}) \Big|_{\{t_{\widetilde{N},i}\}=0} = -\frac{\partial \mathcal{F}_0}{\partial t_i}$$

4.3 Perturbed Correlation Functions

Finally, using the expressions for the gravitational descendants we see that the 2- and 3-point correlation functions of the theory can be easily calculated as follows. Further knowing these functions, all the higher order correlation functions of the theory can also be easily obtained.

1. The 2-point function is given by:

$$< \sigma_M(\phi_i)\sigma_N(\phi_j) > = \left[\sum_{p=0}^k \sum_{q=0}^k \left(\frac{\partial}{\partial t_p} \mathbb{R}^i_{(M)}(\underline{t}) \right) \left(\frac{\partial}{\partial t_q} \mathbb{R}^j_{(N)}(\underline{t}) \right) \right] < \phi_{k-p}\phi_{k-q} >$$

$$= \left[\sum_{p=0}^k \sum_{q=0}^k \left(\frac{\partial}{\partial t_p} \mathbb{R}^i_{(M)}(\underline{t}) \right) \left(\frac{\partial}{\partial t_q} \mathbb{R}^j_{(N)}(\underline{t}) \right) \right] \delta_{p+q,k}$$

Thus we have:

$$< \boldsymbol{\sigma_M(\phi_i)\sigma_N(\phi_j)} > = \left[\sum_{p=0}^k \left(\frac{\partial}{\partial t_p} \mathbb{R}^i_{(M)}(\underline{t}) \right) \left(\frac{\partial}{\partial t_{k-p}} \mathbb{R}^j_{(N)}(\underline{t}) \right) \right] \quad (4.\ 17)$$

Factorization: It is known from general theorems of topological theories that the correlation functions satisfy the *factorization* hypothesis. We can easily check this explicitly from our construction as follows. The 2-point function satisfies the important factorization property as can now be easily verified. Using the result (3. 20), we can rewrite the expression for the 2-point function as:

$$< \sigma_M(\phi_i)\sigma_N(\phi_j) > = \left[\sum_{p=0}^k \left(\frac{\partial}{\partial t_p} \mathbb{R}^i_{(M)}(\underline{t}) \right) \left(\frac{\partial}{\partial t_{k-p}} \mathbb{R}^j_{(N)}(\underline{t}) \right) \right]$$

$$= \sum_{p=0}^k < \sigma_M(\phi_i)\,\phi_p > < \sigma_N(\phi_j)\,\phi_{k-p} >$$

Since $\phi^p = \eta_{(0)}^{pm}\phi_m = \delta^{p+m,k}\phi_m = \phi_{k-p}$, we have the nice *factorization* property:

$$< \sigma_M(\phi_i)\sigma_N(\phi_j) > = \sum_{m,l=0}^k < \sigma_M(\phi_i)\,\phi_m > \eta_{(0)}^{ml} < \phi_l\,\sigma_N(\phi_j) > \quad (4.\ 18)$$

2. Similarly, the expression for the 3-point function becomes:

$$
\begin{aligned}
< \sigma_M(\phi_i)\sigma_N(\phi_j)\sigma_P(\phi_l) > \quad = \quad & \sum_{\{p,q,r\}=0}^{k} \left[\left(\frac{\partial \, \mathbb{R}^i_{(M)}(\underline{t})}{\partial t_{k-p}} \right) \left(\frac{\partial \, \mathbb{R}^j_{(N)}(\underline{t})}{\partial t_{k-q}} \right) \left(\frac{\partial \, \mathbb{R}^l_{(P)}(\underline{t})}{\partial t_{k-r}} \right) \right] \\
& \qquad\qquad\qquad\qquad\qquad \cdot < \phi_p \phi_q \phi_r > \\
= \quad & \sum_{\{p,q,r\}=0}^{k} \left[\left(\frac{\partial \, \mathbb{R}^i_{(M)}(\underline{t})}{\partial t_{k-p}} \right) \left(\frac{\partial \, \mathbb{R}^j_{(N)}(\underline{t})}{\partial t_{k-q}} \right) \left(\frac{\partial \, \mathbb{R}^l_{(P)}(\underline{t})}{\partial t_{k-r}} \right) \right] \\
& \qquad\qquad\qquad \cdot \frac{\partial_p \partial_q}{(r+1)(k+r+3)} \oint dx \, L^{k+r+3}
\end{aligned}
$$

Therefore the expression for the 3-point function is:

$$
< \sigma_M(\phi_i)\sigma_N(\phi_j)\sigma_P(\phi_l) > \quad =
$$

$$
- \sum_{\{p,q,r\}=0}^{k} \left[\left(\frac{\partial \, \mathbb{R}^i_{(M)}(\underline{t})}{\partial t_{k-p}} \right) \left(\frac{\partial \, \mathbb{R}^j_{(N)}(\underline{t})}{\partial t_{k-q}} \right) \left(\frac{\partial \, \mathbb{R}^l_{(P)}(\underline{t})}{\partial t_{k-r}} \right) \right] \cdot \left(\frac{\partial^2}{\partial t_p \partial t_q} \, \mathbb{R}^r_{(1)}(\underline{t}) \right)
$$

$$
(4. \ 19)
$$

Setting the indices $M = N = P = 0$ in the above then gives us back the expressions well known [3] for the gravity-free theory.

5 Recursion Relations for Correlation Functions

In this section, we shall prove some exact relations for correlation functions in our gravity-coupled model. The nice thing about these relations is that these will be valid in the *entire* phase space of our theory (*i.e.*, will be valid on the large phase space as well). When reformulated in the language of path-integrals, these relations then become the generalized Ward identitites for the theory. Examples of such relations (like the puncture equation) valid on the *small phase space* have been used in the discussions of topological gravity [8], [9], and we now try to extend such results to the *large phase space*. As a by-product, we also rederive, in an alternative fashion, the known results (on the *small phase space*). We show the calculations in some details as some highly non-trivial manipulations are involved in the intermediate stages.

5.1 Ward Identity

From the definition [11] of correlation functions in *genus-* 0, we get:

$$
\begin{aligned}
\frac{\partial}{\partial t_{N,i}} < \sigma_M(\phi_j) > \quad = \quad & \frac{\partial}{\partial t_{N,i}} \oint dx \left(\frac{\sigma_M(\phi_j)}{W'} \right) \\
= \quad & - \oint dx \left(\frac{1}{W'} \right) \partial_x \left[\frac{\sigma_M(\phi_j) \, \sigma_N(\phi_i)}{W'} \right] \\
& - \oint dx \, \frac{\sigma_N(\phi_i)}{W'} \partial_x \left[\frac{\sigma_M(\phi_j)}{W'} \right] - \oint dx \, \frac{\sigma_N(\phi_i)\sigma_M(\phi_j)}{W'} \partial_x \left(\frac{1}{W'} \right)
\end{aligned}
$$

(where we have used eq. (4. 1) to replace the first term,

and have also subsequently neglected the + subscript)

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$$= -\oint dx \left(\frac{\sigma_N(\phi_i)}{W'}\right) \partial_x \left[\frac{\sigma_M(\phi_j)}{W'}\right] \qquad (5.\ 1)$$

$$= -<\sigma_N(\phi_i)\,\sigma_{M-1}(\phi_j)> + \sum_{l=0}^{k}\left(\frac{\partial}{\partial t_l}\mathbb{R}^j_{(M)}(\underline{t})\right)\oint\left(\frac{\phi_{k-l}}{W'}\right)\sigma_{N-1}(\phi_i)$$

$$= -<\sigma_N(\phi_i)\,\sigma_{M-1}(\phi_j)> + <\sigma_{N-1}(\phi_i)\,\sigma_M(\phi_j)>$$

In the intermediate steps, we have often traded the $[\dots]_+$ contribution of certain terms in favour of the entire (*i.e.*, without the +subscript) term. However, in reality, no harm is done as the contribution from the terms thus added or discarded actually *vanish*. Thus for example in one of the intermediate stages of our calculation, we have thrown off the term: $\oint dx \left(\frac{1}{W'}\right)\partial_x\left[\frac{\sigma_M(\phi_j)\,\sigma_N(\phi_i)}{W'}\right]_-$. However, since $\frac{1}{W'} \sim \frac{1}{x} + \frac{1}{x^2} + \dots$, one can easily see that the contribution from the above term *vanishes*. Similarly, we have also neglected the term: $\oint dx \left(\frac{\phi_{k-l}}{W'}\right)\partial_x\left[\left(\frac{\sigma_N(\phi_i)}{W'}\right)\right]_-$. Once again, since $\left(\frac{\phi_{k-l}}{W'}\right) \sim \frac{1}{x} + \dots$ and this implies that the contribution from this term also *vanishes*. Hence we have the following important equation:

$$\frac{\partial}{\partial t_{N,i}} <\sigma_M(\phi_j)> = -<\sigma_{M-1}(\phi_j)\,\sigma_N(\phi_i)> + <\sigma_M(\phi_j)\,\sigma_{N-1}(\phi_i)> \quad (5.\ 2)$$

We note that in the path-integral formulation, the operation
of $\left(-\frac{\partial}{\partial t_{N,i}}\right)$ corresponds, in the *lowest* order, to the $\sigma_N(\phi_i)$ operator insertions. The above equation shows how the expectaion value of $\sigma_M(\phi_j)$ depends on the coupling $t_{N,i}$. From the above equation, we may now make the following conclusions.

5.1.1 Some Special Cases

Setting specific values to the indices N, M, i, j in the above equation (5. 2) (and always remembering the constraint: $\sigma_{-1}(\phi_j) \equiv 0$), we get the following conclusions:

1. $\boxed{N = 0}$: In this case the above equation tells us:

$$\frac{\partial}{\partial t_{0,i}} <\sigma_M(\phi_j)> = -<\sigma_{M-1}(\phi_j)\,\sigma_0(\phi_i)>$$

$$\implies \frac{\partial}{\partial t_i} <\sigma_M(\phi_j)> = -<\sigma_{M-1}(\phi_j)\,\phi_i>$$

$$= -\frac{\partial}{\partial t_i}\mathbb{R}^j_{(M-1)}(\underline{t})$$

Thus we once again arrive at the equation:

$$\frac{\partial}{\partial t_i}\mathbb{R}^j_{(N)}(\underline{t}) = <\sigma_N(\phi_j)\,\phi_i>$$

which is exactly the same (3. 20) as derived earlier.

We also have the useful equation:

$$\frac{\partial}{\partial t_i} <\sigma_N(\phi_j)> = -<\sigma_{N-1}(\phi_j)\,\phi_i> \qquad (5.\ 3)$$

Also in this case, setting the index $i = 0$, we get the *puncture equation*:

$$\frac{\partial}{\partial t_0} < \sigma_N(\phi_j) > = - < \sigma_{N-1}(\phi_j)\,\phi_0 > = - < \sigma_{N-1}(\phi_j)\,\mathbf{P} > \qquad (5.\ 4)$$

which is the *quantum version* of the operator equation $\frac{\partial}{\partial t_0}\sigma_N(\phi_j) = -\sigma_{N-1}(\phi_j)$ derived earlier. Further the above equation also tells us that in the special case when all the couplings to the gravitational descendants vanish and the *puncture operator* reduces to the *identity* operator:

$$\frac{\partial}{\partial t_0} < \sigma_{N+1}(\phi_i) > \Big|_{\{t_{\widetilde{N},i}\} = 0} = - < \sigma_N(\phi_i) >$$

$$\implies \frac{\partial}{\partial t_0}\,\mathbb{R}^i_{(N)}(\underline{t})\Big|_{\{t_{\widetilde{N},i}\} = 0} = -\,\mathbb{R}^i_{(N-1)}(\underline{t})\Big|_{\{t_{\widetilde{N},i}\} = 0} \qquad (5.\ 5)$$

which is exactly the same as derived earlier from a different standpoint.

2. $\boxed{M = 0}$: In this case that above equation tells us:

$$\frac{\partial}{\partial t_{N,i}} < \phi_j > = < \sigma_{N-1}(\phi_i)\,\phi_j >$$

$$\implies \frac{\partial}{\partial t_{1,i}} < \phi_j > = < \sigma_0(\phi_i)\,\phi_j >$$

$$= < \phi_i\,\phi_j >$$

$$= \delta_{i+j,k}$$

So that inverting the above equation gives:

$$< \phi_j > = \int dt_{1,i}\,\delta_{i+j,k}$$

$$= t_{1,k-j}$$

giving us the result:

$$t_{1,i} = < \phi_{k-i} > \qquad (5.\ 6)$$

which gives us an explicit expression for the couplings to the lowest order gravitational descendants.

3. $\boxed{N = 1}$: In this case the eq.(5. 2) reduces to:

$$\frac{\partial}{\partial t_{1,i}} < \sigma_M(\phi_j) > = - < \sigma_{M-1}(\phi_j)\,\sigma_1(\phi_i) > + < \sigma_M(\phi_j)\,\phi_i > \qquad (5.\ 7)$$

so that setting $i = 0$, gives us the *dilaton equation*:

$$\frac{\partial}{\partial t_{1,0}} < \sigma_M(\phi_j) > = - < \sigma_{M-1}(\phi_j)\,\Phi_0 > + < \sigma_M(\phi_j)\,\mathbf{P} > \qquad (5.\ 8)$$

where $\sigma_1(\phi_0) = \Phi_0$ is the dilaton operator and t_0 is the corresponding coupling and the above equation shows how the dilaton and the puncture operators get inserted into the correlation functions.

5.2 Further Generalizations

Now that we have obtained the equation for $\frac{\partial}{\partial t_{N,i}} < \sigma_M(\phi_j) >$, we may proceed towards obtaining the similar expressions for the quantities like $\frac{\partial}{\partial t_{N,i}} < \sigma_M(\phi_j)\sigma_L(\phi_m) >$, $\frac{\partial}{\partial t_{N,i}} < \sigma_M(\phi_j)\sigma_L(\phi_m)\sigma_P(\phi_l) >, etc.$ and the higher order correlation functions. However, instead of going through similar computational procedures, we may take an alternative shortcut route as follows.

We have already seen that the quantities $\mathbb{R}^l_{(N)}(\underline{t})$ satisfy the important equation (4. 13). Using the result (3. 20), we get

$$\frac{\partial^2}{\partial t_i \partial t_j} \, \mathbb{R}^l_{(N)}(\underline{t}) \quad = \quad \frac{\partial}{\partial t_j}\left[< \sigma_N(\phi_l)\,\phi_i > \right]$$

Substituting the above results into the previous equation we get the identity:

$$\frac{\partial}{\partial t_j} < \sigma_N(\phi_l)\,\phi_i > \quad = \quad -\sum_{m=0}^{k} \mathcal{C}_{ij}{}^m < \sigma_{N-1}(\phi_l)\,\phi_m > \tag{5. 9}$$

We have already seen that the 2-point function *factorizes* nicely. Using this wisdom we then have:

$$\frac{\partial}{\partial t_l} < \sigma_N(\phi_i)\,\sigma_M(\phi_j) > \quad = \quad \sum_{s,n} \frac{\partial}{\partial t_l}\left[< \sigma_N(\phi_i)\,\phi_s > \eta^{sn}_{(0)} < \phi_n\,\sigma_M(\phi_j) > \right]$$

$$= \quad \sum_{s,n} \eta^{sn}_{(0)} < \sigma_N(\phi_i)\,\phi_s > \frac{\partial}{\partial t_l} < \sigma_M(\phi_j)\,\phi_n >$$

Using $\eta^{pm}_{(0)}\phi_m = \phi_{k-p}$, and the result from eq.(5. 9), we finally have the following useful *generalization* of our recursion relation to the case of the 2-point function:

$$\frac{\partial}{\partial t_l} < \sigma_N(\phi_i)\,\sigma_M(\phi_j) > \quad = \quad -\sum_{m,n} \mathcal{C}_{ln}{}^m \Bigg[< \sigma_N(\phi_i)\,\phi_{k-n} >< \sigma_{M-1}(\phi_j)\,\phi_m >$$

$$+ < \sigma_M(\phi_j)\,\phi_{k-n} >< \sigma_{N-1}(\phi_i)\,\phi_m > \Bigg] \tag{5. 10}$$

We may now make the following conclusions from the above equation.

1. Setting $M = N = 0$, in the above equation we get:

$$\frac{\partial}{\partial t_l} < \phi_i\phi_j > \quad = \quad 0$$

which just reminds us that the 2-point function is $\underline{t}$- independent.

2. Setting the indices $M = 0$, and $N = 1$ in the identity (5. 9), we have:

$$\frac{\partial}{\partial t_l} < \sigma_1(\phi_i)\,\phi_j > \quad = \quad -\sum_{m=0}^{k} \mathcal{C}_{lj}{}^m < \phi_i\,\phi_m >$$

$$= \quad -\mathcal{C}_{lj}{}^{k-i} \tag{5. 11}$$

But $< \sigma_1(\phi_i)\,\phi_j > = \frac{\partial}{\partial t_j}\mathbb{R}^i_{(1)}(\underline{t})$, which when combined with the above equation gives us the result

$$\left(\frac{\partial^2}{\partial t_j \partial t_l}\right)\mathbb{R}^l_{(N)}(\underline{t}) \quad = \quad -\mathcal{C}_{lj}{}^{k-i}$$

a result that we had also obtained earlier (See Corollary Sect.**4**).

3. Setting $l = 0$ in eq.(5. 10), we get:

$$
\begin{aligned}
\left(-\frac{\partial}{\partial t_0}\right) < \sigma_N(\phi_i)\,\sigma_M(\phi_j) > \quad &= \quad \sum_m \Bigg[< \sigma_N(\phi_i)\,\phi^m > < \sigma_{M-1}(\phi_j)\,\phi_m > \\
&\quad + \; < \sigma_M(\phi_j)\,\phi^m > < \sigma_{N-1}(\phi_i)\,\phi_m > \Bigg] \\
&= \quad \Bigg[< \sigma_N(\phi_i)\,\sigma_{M-1}(\phi_j) > \\
&\quad + \; < \sigma_{N-1}(\phi_i)\,\sigma_M(\phi_j) > \Bigg] \qquad (5.\ 12)
\end{aligned}
$$

which is a generalized form of the *puncture equation.*

Thus working along similar lines (*i.e.* proceeding with computations like those leading to (5. 2) and (5. 10) as previously shown), one may then prove the following generalized operator form of the *puncture equation*:

$$\left(-\frac{\partial}{\partial t_0}\right) < \prod_j \sigma_{N_j}\!\left(\phi_{n_j}\right) > \quad = \quad \sum_l < \sigma_{N_l-1}(\phi_{n_l})\prod_{j\neq l}\sigma_{N_j}\!\left(\phi_{n_j}\right) > \qquad (5.\ 13)$$

Thus we have an alternative operator method of derivation of the puncture equation which has been previously ([3, 8, 9, 16]) derived using an entirely different procedure. Finally, we can make an important observation here. In the next section we shall see that the coordinates in the large phase space can eventually be expressed in terms of those in the small phase space. Thus having done this, we can easily generalize the recursion relations like (5. 10) to the entire phase space.

6 Relation between $t_{N,i}$ and $t_{0,i}$

Finally, in this section we focus our attention to the central conclusion of our present work. We investigate the properties of the coordinates of the *large phase space* of our gravity-coupled model and we try to *determine* their explicit forms. We may recall that we began by considering the minimal action perturbed by *all* possible fields belonging to the family of gravitational descendants of the primary fields in our theory. These perturbing fields couple by *a priori arbitrary* couplings $t_{N,i}$ (the "coordinates" on the *large phase space*). These couplings are initially completely unconstrained apart from the demand of reparametrization invariance. However, in the gravity-free case, it is known that due to certain underlying general properties of TCFT, these couplings can not just be anything arbitrary, but are highly constrained. In fact, these have to be necessarily completely determined functions (see eqn. (2. 10)) of the *flat-coordinates* – and are hence are no longer arbitrary at the end of the day. In this case,

the fact that the 2-point function (which also defines the metric in this case) of the theory remains unaltered even in the presence of perturbations leads to the explicit determination of all the couplings (the "coordinates" of the *small phase space*) to the perturbing chiral primaries of the theory. After coupling this model to gravity, we then have a much enlarged (infinite-dimensional) phase space. In the existing literature, not much has been said about the properties of these couplings to the gravitational descendants. In particular, with gravity switched on we have an infinite number of *a priori arbitrary* couplings (to the gravitational descendants) in the theory, and it is interesting and quite relevant to ask ourselves *what* we can say about these couplings, and can we *determine* these couplings explicitly, just as we did in the gravity-free case.

Our analysis in this section will seek to obtain the answers for this question in the affirmative. In fact we shall see that the answer to both these questions is *yes*. The underlying topological structure of the gravity-coupled model *does*, in this case, also lead to the complete and explicit (modulo constant translations) determination of *all* the couplings to the gravitational descendants of the theory. We shall see that one can determine these infinite set of couplings in the form of known (but infinite in number) functions of the coordinates in the small phase space. Further, the coordinates in the small phase space are, in turn, known in terms of the flat coordinates. Thus eventually, we get to determine all the couplings explicitly.

Let us now try to obtain this relation between the perturbation parameters $(t_{N,i})$ (3. 15) of the general gravity-coupled model and those $(t_{0,i} \equiv t_i)$ when the gravity is *switched off*. The couplings to the chiral primaries (*i.e.* our flat coordinates in the *small phase space*) are already known, being given by (2. 10). So the remaining task is to solve for the couplings to the gravitational descendants. We argue that the couplings to the gravitational descendants *can* also be determined completely using the informations we already have. Once we have done this, it means that now the entire phase space of the theory is known.

We shall achieve this by trying to evaluate the 2-point function $< \sigma_M(\phi_i)\sigma_N(\phi_j) >$ of the general gravity-coupled model in two different ways and then comparing the results to obtain the required relation. One way of obtaing the expression for the above correlation function has already been discussed in Sect. 4.3. We now evaluate the same in an alternative method, without resorting to the *reduction formula* (3. 13) for the gravitational descendants. Thus following the prescription of *Vafa* [11] we have (the details of the derivation (6. 1) are in the appendix):

$$< \sigma_M(\phi_i)\sigma_N(\phi_j) > \quad = \quad \frac{\partial}{\partial t_{N,j}} \mathbb{R}^i_{(M)}(\underline{t}) \quad + \quad \frac{\partial}{\partial t_{N-1,j}} \mathbb{R}^i_{(M+1)}(\underline{t}) \qquad (6.\ 1)$$

Now writing

$$\frac{\partial}{\partial t_l} \quad = \quad \sum_{N,j} \left(\frac{\partial t_{N,j}}{\partial t_l}\right) \frac{\partial}{\partial t_{N,j}}$$

$$= \quad \sum_{N,j} \left(\mathbb{D}^{(N)}\right)_{jl} \left(\frac{\partial}{\partial t_{N,j}}\right)$$

where we have introduced N (where $N = 0, 1, 2, \ldots \infty$) matrices (each of size $k \times k$) $\mathbb{D}^{(N)}$ defined by:

$$\mathbb{D}^{(N)}_{ij} \quad \overset{\text{def}}{=} \quad \left(\frac{\partial t_{N,i}}{\partial t_j}\right) \quad \equiv \quad \left(\frac{\partial t_{N,i}}{\partial t_{0,j}}\right) \qquad (0 \leq i, j \leq k) \qquad (6.\ 2)$$

so that we may write:

$$\frac{\partial}{\partial t_{N,j}} \quad = \quad \sum_l \left(\mathbb{D}^{(N)}\right)^{-1}_{lj} \cdot \frac{\partial}{\partial t_l}$$

Thus, *determining* the matrices $\mathbb{D}^{(N)}$ explicitly amounts to *knowing*[6] *the couplings* $t_{N,i}$ in terms of t_i; and as the latter are already known in terms of the superpotential, this information is sufficient to determine *all* the couplings. Clearly we have:

$$\mathbb{D}^{(0)}_{ij} \quad = \quad \delta_{ij}$$

Using these matrices introduced above, we can recast our expression for the 2-point function as:

$$< \sigma_M(\phi_i)\sigma_N(\phi_j) > \quad = \quad \sum_l \left(\mathbb{D}^{(N)}\right)^{-1}_{lj} \frac{\partial}{\partial t_l} \mathbb{R}^i_{(M)}(\underline{t})$$

$$+ \quad \sum_l \left(\mathbb{D}^{(N-1)}\right)^{-1}_{lj} \frac{\partial}{\partial t_l} \mathbb{R}^i_{(M+1)}(\underline{t}) \qquad (6.\ 3)$$

Comparing eqs.(3.27) and (3.37) we can then have the following equation:

$$-\sum_l \left[\left(\mathbb{D}^{(N)}\right)^{-1}_{lj} - \left(\frac{\partial \mathbb{R}^j_{(N)}}{\partial t_{k-l}}\right) \right] \left(\frac{\partial}{\partial t_l} \mathbb{R}^i_{(M)}(\underline{t}) \right)$$

$$= \quad \sum_l \left(\mathbb{D}^{(N-1)}\right)^{-1}_{lj} \left(\frac{\partial}{\partial t_l} \mathbb{R}^i_{(M)}(\underline{t}) \right) (6.\ 4)$$

which may be in a more convenient notation[7] as:

$$\left[\left(\frac{\partial}{\partial t_l} \mathbb{R}^i_{(M+1)}(\underline{t}) \right) \left(\frac{\partial}{\partial t_{k-l}} \mathbb{R}^j_{(N-1)}(\underline{t}) \right) \right] \quad = \quad -\Delta^{(N)}_{lj} \left(\frac{\partial}{\partial t_l} \mathbb{R}^i_{(M)}(\underline{t}) \right)$$

$$- \Delta^{(N-1)}_{lj} \left(\frac{\partial}{\partial t_l} \mathbb{R}^i_{(M+1)}(\underline{t}) \right)$$

$$\text{where} \quad \Delta^{(N)}_{lj} \quad \overset{\text{def}}{=} \quad \left[\left(\mathbb{D}^{(N)}\right)^{-1}_{lj} - \left(\frac{\partial \mathbb{R}^j_{(N)}}{\partial t_{k-l}}\right) \right]$$

$$\text{(And clearly,} \quad \Delta^{(0)}_{lj} \quad = \quad 0.)$$

And finally, introducing the operators $\Omega^{(N)}_j$ defined by:

$$\Omega^{(N)}_j(\underline{t}) \quad \overset{\text{def}}{=} \quad \sum_l \Delta^{(N)}_{lj} \frac{\partial}{\partial t_l}$$

we may write the *equation determining the relation between the couplings*:

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[6]Having done this, the remaining task is quite easy, as we only need to solve these first-order differential equations — to eventually express the couplings $t_{N,i}$s in terms of the t_is.

[7]The summation convention is assumed over the repeated small indices i, l, *etc.*

$$\sum_l \left[\left(\frac{\partial}{\partial t_l} \mathbb{R}^i_{(M+1)}(\underline{t}) \right) \left(\frac{\partial}{\partial t_{k-l}} \mathbb{R}^j_{(N-1)}(\underline{t}) \right) \right] = - \left[\Omega^{(N)}_j \mathbb{R}^i_{(M)}(\underline{t}) \right.$$
$$\left. + \Omega^{(N-1)}_j \mathbb{R}^i_{(M+1)}(\underline{t}) \right] \quad (6.\ 5)$$

Since $\Omega^{(0)}_j = 0$, the above equation can therefore be used to recursively determine all the $\Omega^{(N)}_j$s and hence finally the N matrices $\mathbb{D}^{(N)}-$ and hence eventually, the desired relationship. Even though the above equation does not look very friendly at first sight, we can in fact solve it setting specific values for N and obtain the expressions for the quantities $\mathbb{D}^{(N)}$. To make matters simple, we may without any loss of generality, set $M = 0$ in the above equation which then reduces[8] to the following simplified form:

$$\left[\Delta^{(N)}_{ij} + \Omega^{(N-1)}_j \mathbb{R}^{k-i}_{(1)} \right] + \sum_l \left[\left(\frac{\partial}{\partial t_l} \mathbb{R}^{k-i}_{(1)} \right) \left(\frac{\partial}{\partial t_{k-l}} \mathbb{R}^j_{(N-1)} \right) \right] = 0 \quad (6.\ 6)$$

with the additional constraints: $\Delta^{(0)} = \Omega^{(0)} = 0$

- **Note:** If further we sum both sides of eq. (6. 5) over $M = -\infty$ to ∞, we can write the above equation in terms of the quantity Ψ defined in (4. 14), as:

$$\left[\Omega^{(N)}_j + \Omega^{(N-1)}_j \right] \Psi^{(i)}(\underline{t})$$
$$= - \left[\left(\frac{\partial}{\partial t_l} \Psi^{(i)} \right) \left(\frac{\partial}{\partial t_{k-l}} \mathbb{R}^j_{(N-1)}(\underline{t}) \right) \right] \quad (6.\ 7)$$

which we may rewrite in a more convenient form as:

$$\sum_l \left[\Delta^{(N)}_{lj} + \Delta^{(N-1)}_{lj} + \left(\frac{\partial}{\partial t_{k-l}} \mathbb{R}^j_{(N-1)}(\underline{t}) \right) \right] \left(\frac{\partial}{\partial t_l} \Psi^{(i)} \right) = 0 \quad (6.\ 8)$$

which can be interpreted as a sort of a *flow* equation in the space of couplings (the small phase space).

Thus we see that even in the presence of gravity, one can say quite a lot about the corresponding couplings. The couplings to the gravitational descendants cannot just be anything arbitrary, but are highly constrained by the underlying topological symmetries of the theory. The basic symmetries of the theory are stringent enough to completely determine the functional form of all these couplings. Since the couplings to the descendant fields are now

---

[8]where we have used the fact that $\Omega^{(N)}_j \mathbb{R}^{k-i}_{(0)}(\underline{t}) = \Delta^{(N)}_{ij}$

determined in terms of those to the chiral primaries and these in turn in terms of the *flat coordinates*, we see eventually that all the informations about the large phase space can be extracted from the superpotential characterising the matter sector of the theory. Even though the above observation is based on our calculations with a specific kind of matter model, we believe that these conclusions should hold true quite generally (*i.e.* for other models as well). This therefore lends credence to the view that $2d$ topological gravity is indeed an *induced* effect.

Some Special Cases: Let us now try to solve our master equation explicitly for some special cases to show that this formalism indeed provides us with the necessary framework for determining the coordinates of the large phase space. As our formalism is of recursive nature, we need to proceed from the couplings to the lowest order gravitational descendants. Setting $N = 1$ in the equation (6. 6), we get

$$\Delta_{ij}^{(1)} + \sum_l \left(\frac{\partial}{\partial t_l} \mathbb{R}_{(1)}^{k-i}\right)\left(\frac{\partial}{\partial t_{k-l}} \mathbb{R}_{(0)}^j\right) = 0$$

$$\implies \Delta_{ij}^{(1)} = -\left(\frac{\partial}{\partial t_j} \mathbb{R}_{(1)}^{k-i}(\underline{t})\right) \tag{6. 9}$$

This eventually gives us:

$$\left(\mathbb{D}^{(1)}\right)_{ij}^{-1} = \sum_l \left[\left(\frac{\partial}{\partial t_l} \mathbb{R}_{(1)}^j\right) - \left(\frac{\partial}{\partial t_j} \mathbb{R}_{(1)}^l\right)\right](\delta_{i+l,k}) \tag{6. 10}$$

$$= \sum_l e_{mlj}\, \delta_{i+l,k} \left(\overrightarrow{\nabla}_{\underline{t}} \times \overrightarrow{\mathbb{R}}_{(1)}\right)_m \tag{6. 11}$$

(where $\overrightarrow{\mathbb{R}}_{(1)} = \{\mathbb{R}_{(1)}^i\}$ is interpreted as a vector in the $(k+1)$-dimensional space of chiral primaries, and the curl is taken with respect to the couplings t_i treated as coordinates *i.e.* $\overrightarrow{t} = \{t_i\}$) from which the desired relation between $t_{1,i}$ and t_i may be obtained. We may then put back the obtained values in the equation to solve for the next order couplings. Thus recursively the entire family of couplings $\{t_{N,i}\}$ can be determined.

Similarly, setting $N = 2$ in the equation (6. 6), and using the previous results we can obtain the following result:

$$\Delta_{ij}^{(2)} = \sum_l \left(\frac{\partial \mathbb{R}_{(1)}^{k-i}}{\partial t_{k-l}}\right)\left[\frac{\partial \mathbb{R}_{(1)}^l}{\partial t_j} - \frac{\partial \mathbb{R}_{(1)}^j}{\partial t_l}\right] \tag{6. 12}$$

Thus introducing the antisymmetric tensor $V_{ij}^{(N)}$ defined by

$$V_{ij}^{(N)} \stackrel{\text{def}}{=} \left[\frac{\partial \mathbb{R}_{(N)}^i}{\partial t_j} - \frac{\partial \mathbb{R}_{(N)}^j}{\partial t_i}\right] \tag{6. 13}$$

(which trivially satisfies $\partial_l V_{ij}^{(N)} + cyclic\ perm. = 0$), we see that the final expressions for the matrices $\left(\left[\mathbb{D}^{(N)}\right]^{-1}\right)_{ij}$ can be written more compactly as:

$$\left(\left[\mathbb{D}^{(1)} \right]^{-1} \right)_{ij} \;\; = \;\; \sum_l \left(\frac{\partial \, \mathbb{R}_{(0)}^{k-i}}{\partial t_{k-l}} \right) V_{jl}^{(1)} \;\; = \;\; \sum_l \delta_{i+l,k} \, V_{jl}^{(1)} \qquad\qquad (6.\ 14)$$

$$\left(\left[\mathbb{D}^{(2)} \right]^{-1} \right)_{ij} \;\; = \;\; \left(\frac{\partial \, \mathbb{R}_{(2)}^{j}}{\partial t_{k-i}} \right) + \sum_l \left(\frac{\partial \, \mathbb{R}_{(1)}^{k-i}}{\partial t_{k-l}} \right) V_{jl}^{(1)} \qquad\qquad (6.\ 15)$$

$$\left(\left[\mathbb{D}^{(3)} \right]^{-1} \right)_{ij} \;\; = \;\; \left(\frac{\partial \, \mathbb{R}_{(3)}^{j}}{\partial t_{k-i}} \right) - \sum_l \left(\frac{\partial \, \mathbb{R}_{(1)}^{k-i}}{\partial t_l} \right) \left(\frac{\partial \, \mathbb{R}_{(2)}^{j}}{\partial t_{k-l}} \right) + \sum_l \left(\frac{\partial \, \mathbb{R}_{(1)}^{k-i}}{\partial t_{k-l}} \right) V_{jl}^{(1)} \; (6.\ 16)$$

etc.

The knowledge of the matrix elements $\left(\left[\mathbb{D}^{(N)} \right]^{-1} \right)_{ij}$ allows us to obtain the expressions for the couplings $t_{N,i}$ in terms of the coordinates t_i of the small phase space. The latter are in turn obtainable in terms of the superpotential W characterizing the topological matter sector (the A_{k+1} superpotential in our case) of the theory. Thus though we start off with an infinite dimensional coupling constant space, eventually the entire phase space of couplings can be expressed as known polynomial functions of the flat-coordinates based on the information about the superpotential of the matter sector *only* — *i.e.* we are thus able, at the end of the day, to explicitly determine the functional forms $t_{N,i}(t_0, t_1, t_2, \ldots, t_k)$ of the gravitational descendant couplings.

7   Discussions and Further Outlook

In our formalism, we have effectively considered the topological minimal model A_{k+1} in a *background* of gravitational descendant fields. These descendant fields have relative strengths determined by the couplings $t_{N,i}$. One can thus interpret this enlarged phase space of the theory as the space of all perturbations of our original model, with the couplings playing the role of coordinates in the space of perturbations. Further, the constraint of *flatness* on the *coordinates* of the small phase space uniquely fixes up the couplings to the chiral primaries. We have also seen, how by computing the 2-point correlation functions in two different ways, namely, once working on the large phase space, and then secondly using the reduction formula for the gravitational descendants (and hence effectively working on the small phase space, renormalized by the gravity-couplings), and finally comparing the two results, one can also relate the coordinates on the large phase space to those on the small phase space. In effect, this tells us that we have succeeded in reducing the infinite number of *a priori arbitrary* couplings $t_{N,i}$, to an infinite number of *known* functions of a *finite* number of variables t_i.

Using the properties of the underlying multiplicative ring structure of the $N = 2$ theory of our matter sector, we have obtained a set of orthogonal polynomials. (These are orthogonal with respect to the inner product on the ring). To lowest order, these *coincide* with the basis of the perturbed chiral ring, but in higher orders this convenient property is destroyed as each of these polynomials receive contributions from the polynomials of a different order, coupled by the coordinates on the large phase space. Thus in effect these polynomials get renormalized at each order, and by imposing the condition of orthogonality at each order,

one can then get an interesting hierarchy of differential equations, governing the coordinate-dependences of these polynomials. In obtaining such relations, we have found an interesting operator equivalence (4. 10), which allows us to directly write down such equations at any order. Further, this operator equivalence may have deeper mathematical interpretaions in along the lines of Saito's work [14, 15].

The operator versions of the relations determining the *gravitaional dressing* of the correlation functions that we have obtained in Sect.5, are essentialy generalized Ward identities in the language of path-integrals, and valid throughout the large phase space. We have, in our work, considered only the two lowest order functions, namely the 1- and 2-point correlation functions. Higher order correlation functions can eventually be written down in terms of these (using the factorization property, and the recursion relations), and hence such generalized Ward identities on the large phase space, for any N-point correlation function can now be constructed, thus leading to an alternative derivation of the Virasoro recursion relations of topological gravity. Restricting these identities to the small phase space gives us the generalized *puncture* and *dilaton* equations which have been derived previously from an entirely different standpoint [3, 8, 9, 16].

The generalized puncture and dilaton equations are known to give rise to an interesting non-commutative contact term algebra (isomorphic to the Virasoro algebra) [3, 8, 9]. It would be interesting to investigate how these commutation relations can be obtained in our formailsm. In our formalism, we have viewed the model of topological gravity coupled to topological matter as the theory of a topological minimal matter in the presence of an infinite-dimensional background of gravitational fields. One can then compute the β functions for the couplings to these background fields, and thus investigate the *multi-critical* behaviour of the theory. One could also, alternatively, use our expression for the free-energy to examine the aspects of *gravitational phase transitions* in the theory. Because of the close relationship between the A_{2k+1} and D_{k+2} minimal models, our above formalism can be readily modified to discuss coupling the D_{k+2} model to $2d$ topological gravity. Further we may easily consider the $k \to 0$ limit of our results to derive the results for *pure* gravity. This limit would thus lead to a complete solution of pure topological gravity.

A   Appendix

A.1   Proof of the Property (4.12)

Using eq. (2.38) from the previous section,

$$
\begin{aligned}
b(N,l) \oint dx\, L^{(N-1)(k+2)+l+1} \left(\frac{\partial \phi_i}{\partial t_j} \right) &= -b(N,l) \oint dx\, L^{(N-1)(k+2)+l+1} \, \partial_x \left[\frac{\phi_i \phi_j}{W'} \right]_+ \\
&= b(N-1,l) \oint dx \left(L^{(N-1)(k+2)+l} \partial_x L \right) \left[\frac{\phi_i \phi_j}{W'} \right]_+ \\
&= b(N-1,l) \oint dx \left[\frac{\sum \mathcal{C}_{ij}{}^m \phi_m}{W'} \right]_+ \left(L^{(N-1)(k+2)+l} \partial_x L \right) \\
&= \sum_m \mathcal{C}_{ij}{}^m \left(\frac{\phi_m}{W'} \right)_+ b(N-1,l) \oint dx \left(L^{(N-1)(k+2)+l} \partial_x L \right) \\
&= 0
\end{aligned}
\tag{A. 1}
$$

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where we have used the fact that $\left[\frac{\phi_m}{W'}\right]_+$ is actually independent of x (from eq. (4. 2) proved earlier) to pull it out of the integration.

A.2   Computation of $< \sigma_M(\phi_i)\sigma_N(\phi_j) >$

Following the prescription [11] of *Vafa*, the 2-point function is given by:

$$
\begin{aligned}
< \sigma_M(\phi_i)\sigma_N(\phi_j) > \quad &= \quad \oint dx \left(\frac{\sigma_M(\phi_i)\sigma_N(\phi_j)}{W'}\right) \\
&= \quad b(M,i) \oint dx \frac{\left[L^{M(k+2)+i}\,\partial_x L\right]_+}{L^{k+1}\partial_x L}\left(-L^{k+1}\frac{\partial L}{\partial t_{N,j}}\right) \\
&= \quad b(M,i) \oint dx \frac{\left[L^{M(k+2)+i}\,\partial_x L\right]}{L^{k+1}\partial_x L}\left(-L^{k+1}\frac{\partial L}{\partial t_{N,j}}\right) \\
&\quad - b(M,i) \oint dx \frac{\left[L^{M(k+2)+i}\partial_x L\right]_-}{L^{k+1}\partial_x L}\left(-L^{k+1}\frac{\partial L}{\partial t_{N,j}}\right) \\
&= \quad \frac{\partial}{\partial t_{N,j}}\,\mathbb{R}^i_{(M)}(\underline{t}) \quad + \quad \mathcal{T} \quad\quad\quad\quad\quad\quad (A.\ 2)
\end{aligned}
$$

where we have called the contribution from the second term $\mathcal{T}$, and this is given by:

$$
\begin{aligned}
\mathcal{T} \quad &\overset{\text{def}}{=} \quad b(M,i) \oint dx \frac{\left[L^{M(k+2)+i}\,\partial_x L\right]_-}{L^{k+1}\partial_x L}\left(L^{k+1}\frac{\partial L}{\partial t_{N,j}}\right) \\
&= \quad -b(M,i) \oint dx \left[L^{M(k+2)+i}\,\partial_x L\right]_-\left(\frac{\sigma_N(\phi_j)}{W'}\right) \\
&= \quad -b(M,i) \oint dx \left[L^{M(k+2)+i}\,\partial_x L\right]_-\left[\int^x \sigma_{N-1}(\phi_j) + \sum_m \left(\frac{\phi_{k-m}}{W'}\right)\frac{\partial}{\partial t_m}\mathbb{R}^j_{(N)}(\underline{t})\right] \\
&= \quad -b(M,i) \oint dx \left[L^{M(k+2)+i}\,\partial_x L\right]_-\left[\int^x \sigma_{N-1}(\phi_j)\right] \\
&\quad\quad \text{(since we have the } x\text{- behaviour }\left(\frac{\phi_{k-m}}{W'}\right) \sim \frac{1}{x} + \ \ \ldots \ \ ) \\
&= \quad -b(M+1,i) \oint dx\ \partial_x\left[L^{M(k+2)+i+1}\right]_-\left[\int^x \sigma_{N-1}(\phi_j)\right] \\
&= \quad b(M+1,i) \oint dx \left[L^{M(k+2)+i+1}\right]_-[\sigma_{N-1}(\phi_j)] \quad\quad \text{(by partial integration)} \\
&= \quad -b(M+1,i) \oint dx \left[L^{M(k+2)+i+1}\right]\left(L^{k+1}\frac{\partial L}{\partial t_{N-1,j}}\right) \\
&= \quad \frac{\partial}{\partial t_{N-1,j}}\,\mathbb{R}^i_{(M+1)}(\underline{t}) \quad\quad\quad\quad\quad\quad\quad\quad (A.\ 3)
\end{aligned}
$$

Hence adding up the contributions from both the terms, we get the desired relation (6. 1) used in our earlier section.

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